Diffusion Filters and Wavelets: What can they learn from each other?

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Mathematische Grundlagen in Vision & Grafik
Based on [1] by Weickert, Steidl, Mrázek, Welk, and Brox
two Methods, same Purpose

Noisy Image

Shift invariant soft wavelet shrinkage

Nonlinear diffusion filtering with total variation diffusivity

Can we join the benefits of this two methods?
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• Basic Methods
• Relations for Space-Discrete Diffusion
• Relations for Fully Discrete Diffusion
• Wavelets with Higher Vanishing Moments
• Summary
Introduction

- Removing noise without sacrificing important structures
- Nonlinear strategies
  - Wavelet shrinkage
  - Nonlinear diffusion filtering based on discrete considerations
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• Basic Methods
  – Wavelet Shrinkage
  – Nonlinear Diffusion Filtering

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Wavelet Shrinkage

Three main steps

- **Analysis**
  Transform noisy data to wavelet coefficients

- **Shrinkage**
  Apply shrinkage function with a threshold parameter

- **Synthesis**
  Reconstruct denoised data from shrunken wavelet coefficients
Wavelet Shrinkage

Analysis

• Lowpass filtering for averaging
• Highpass filtering for details
• Downsampling due to Nyquist criteria

Often implemented as a filter bank

\[
y_{\text{low}}[n] = \sum_{k=-\infty}^{\infty} x[k]g[2n - k]
\]

\[
y_{\text{high}}[n] = \sum_{k=-\infty}^{\infty} x[k]h[2n - k]
\]

\[
y_{\text{low}} = (x * g) \downarrow 2
\]

\[
y_{\text{high}} = (x * h) \downarrow 2
\]
Wavelet Shrinkage

Analysis

\[ f = \sum_{i \in \mathbb{Z}} \langle f, \varphi_i^n \rangle \varphi_i^n + \sum_{j = -\infty}^{n} \sum_{i \in \mathbb{Z}} \langle f, \psi_i^j \rangle \psi_i^j \]

Haar Wavelets

\[ \varphi(x) = 1_{[0, \frac{1}{2})} - 1_{[\frac{1}{2}, 1)} \]

\[ \psi(x) = 1_{[0, 1)} \]
Wavelet Shrinkage

Shrinkage by Soft-Thresholding [2]

\[ S_\theta(s) := \begin{cases} 
  s - \theta \text{sgn } s & \text{if } |s| > \theta \\
  0 & \text{if } |s| \leq \theta 
\end{cases} \]

Synthesis

\[ u := \sum_{i \in \mathbb{Z}} \langle f, \phi_i^n \rangle \phi_i^n + \sum_{j=-\infty}^{\infty} \sum_{i \in \mathbb{Z}} S_\theta\left( \langle f, \psi_i^j \rangle \right) \psi_i^j \]
Nonlinear Diffusion Filtering

Start in 1D

Idea

Obtain a family $u(x,t)$ of filtered versions of a signal $f(x)$ by computing

\[ u_t = \left( g(|u_x|) \right) u_x \]

Initial condition $u(x,0) = f(x)$

$x$ .... Space

$t$ .... Time
Nonlinear Diffusion Filtering

\[ g(|u_x|) \] Diffusivity function

Now:

\[ g(|s|) = \frac{1}{|s|} \]

Total Variation (TV) diffusivity

Problem if \(|s|\) close to zero (unboundend):

\[ u_t = \left( \frac{1}{\sqrt{\epsilon^2 + u_x^2}} u_x \right)_x \]

Regularisation
Nonlinear Diffusion Filtering

Discretisation Scheme

By assuming unit (1) distance between neighbouring pixels we get

\[ u_x = \frac{u_{i+1} - u_i}{1 - 0} = u_{i+1} - u_i \]

Lower index denotes spatial spread

Time step size \( \tau \) leads to

\[ u_t = \frac{u_i^{k+1} - u_i^k}{\tau} \]

upper index \( k \) denotes approximate solution at time \( k \tau \)
Nonlinear Diffusion Filtering

Therefore \[ u_t = (g(|u_x|) u_x)_x \]

becomes

\[ \frac{u_{i}^{k+1} - u_{i}^{k}}{\tau} = g(|u_{i+1}^{k} - u_{i}^{k}|) (u_{i+1}^{k} - u_{i}^{k}) - g(|u_{i}^{k} - u_{i-1}^{k}|) (u_{i}^{k} - u_{i-1}^{k}) \]

resolving by the unknown \( u_{i}^{k+1} \) we achieve

\[ u_{i}^{k+1} = u_{i}^{k} - \tau g(|u_{i}^{k} - u_{i+1}^{k}|) (u_{i}^{k} - u_{i+1}^{k}) + \tau g(|u_{i-1}^{k} - u_{i}^{k}|) (u_{i-1}^{k} - u_{i}^{k}) \]
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Equivalence of two pixel signals

Study connections between „soft Haar“ Wavelet Shrinkage and nonlinear diffusion with TV diffusivity

Two pixel signal \((f_0, f_1)\) with respect to scaling function \(\varphi = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\) and wavelet \(\psi = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\) lead to

\[
\langle f, \varphi \rangle = c = \frac{f_0 + f_1}{\sqrt{2}} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{c}{\sqrt{2}}
Equivalence of two pixel signals

Soft thresholding of wavelet coefficient yields

$$S_\theta(d) = \begin{cases} 
    d - \theta \, \text{sgn} \, d & \text{if} \quad |d| > \theta \\
    0 & \text{if} \quad |d| \leq \theta
\end{cases}$$

leading to the filtered signal \((u_0, u_1)\) .... Shrinkage step

\[ u_0(\theta) = \begin{cases} 
    f_0 + \frac{\theta}{\sqrt{2}} \, \text{sgn} \, (f_1 - f_0) & \text{if} \quad \theta < |f_1 - f_0| / \sqrt{2}, \\
    (f_0 + f_1) / 2 & \text{else,}
\end{cases} \]

\[ u_1(\theta) = \begin{cases} 
    f_1 - \frac{\theta}{\sqrt{2}} \, \text{sgn} \, (f_1 - f_0) & \text{if} \quad \theta < |f_1 - f_0| / \sqrt{2}, \\
    (f_0 + f_1) / 2 & \text{else.}
\end{cases} \]

.... Synthesis step
Equivalence of two pixel signals

Now: space discrete TV Diffusion of \((f_0, f_1)\) with grid size 1

Remember the discretisation scheme

\[
\frac{u_i^{k+1} - u_i^k}{\tau} = g(|u_{i+1}^k - u_i^k|) (u_{i+1}^k - u_i^k) - g(|u_i^k - u_{i-1}^k|) (u_i^k - u_{i-1}^k)
\]

This leads to dynamical system of DE

\[
\dot{u}_0 = \frac{u_1 - u_0}{|u_1 - u_0|}, \quad \dot{u}_1 = -\frac{u_1 - u_0}{|u_1 - u_0|},
\]

with initial conditions

\[
u_0(0) = f_0 \quad \text{and} \quad u_1(0) = f_1\]
Equivalence of two pixel signals

Setting \( w(t) := u_1(t) - u_0(t) \) and \( \eta := f_1 - f_0 \)
leads to initial value problem

\[
\dot{w} = -2 \frac{w}{|w|},
\]

\( w(0) = \eta. \)

still in trouble if \( w=0 \)

get rid of by setting

\[
\dot{w} = -2 \text{sgn} \; w,
\]

\( w(0) = \eta \)

and let \( \text{sgn}(w) \) be any value between -1 and 1 if \( w=0 \)
Equivalence of two pixel signals

Solving this DE exactly gives us

\[ u_0(t) = \begin{cases} f_0 + t \, \text{sgn}(f_1 - f_0) & \text{if } t < |f_1 - f_0|/2, \\ (f_0 + f_1)/2 & \text{else}, \end{cases} \]

Looks very similar to what we have seen in case of wavelets

\[ u_1(t) = \begin{cases} f_1 - t \, \text{sgn}(f_1 - f_0) & \text{if } t < |f_1 - f_0|/2, \\ (f_0 + f_1)/2 & \text{else}. \end{cases} \]

The **BIG** thing: 

**Equivalence** of both approaches if threshold

\[ \theta = \sqrt{2t}. \]
Wavelet inspired Scheme for TV Diffusion

Now can we use 2 Pixel equivalence to spread our thoughts up to N Pixels?

Haar Wavelets: 2 Pixel pairs (independent)

Problem: shrinkage is not shift invariant

Coifman, Donoho 1995 „Cycle Spinning“
Wavelet inspired Scheme for TV Diffusion

use this idea to get a TV Diffusion numerical scheme for N Pixel by using solution of 2 Pixel model as a building block

1) perform TV Diffusion with time step size $2\tau$ on all pixel pairs $(u_{2j}, u_{2j+1})$

2) perform TV Diffusion with time step size $2\tau$ on all pixel pairs $(u_{2j-1}, u_{2j})$

3) average both results

First two steps equivalent to translation invariant Soft Haar with threshold $\Phi = 2\sqrt{2\tau}$
Wavelet inspired Scheme for TV Diffusion

Use 2 Pixel analysis to easily derive the following numerical scheme

\[ u_i^{k+1} = u_i^k + \frac{\tau}{h} \operatorname{sgn} (u_{i+1}^k - u_i^k) \min \left( 1, \frac{h}{4\tau} |u_{i+1}^k - u_i^k| \right) \]
\[ - \frac{\tau}{h} \operatorname{sgn} (u_i^k - u_{i-1}^k) \min \left( 1, \frac{h}{4\tau} |u_i^k - u_{i-1}^k| \right) \]

**Attributes**

- stable
- consistent to "pure" TV diffusion filtering

if

\[ t \leq \frac{h}{4} \min(|u_{i+1}^k - u_i^k|, |u_i^k - u_{i-1}^k|) \]
Wavelet inspired Scheme for TV Diffusion

FIGURE 2. (a) **Top left**: Original signal without noise. (b) **Top right**: With additive Gaussian noise, SNR=8 dB. (c) **Bottom left**: Result with two-pixel scheme. SNR = 24.5 dB. (d) **Bottom right**: Result with classical regularised scheme. SNR = 24.6 dB. From [20].
Generalisations to Images

1D -> 2D

Analogy between one and two dimensional cases

-> determine equivalency between TV diffusion and Soft Haar wavelet shrinkage concerning 2*2 pixel blocks

-> use this 4 pixel solution as building blocks for a numerical scheme for 2D TV diffusion

Stable, conditionally consistent and no additional regularisation needed
Generalisations to Images

FIGURE 3. (a) Left: Original image, $93 \times 93$ pixels. (b) Middle: Standard explicit scheme for regularised TV diffusion ($\varepsilon = 0.01$, $\tau = 0.0025$, 10000 iterations). (c) Right: Same with four-pixel scheme without regularisation ($\tau = 0.1$, 250 iterations). Note that 40 times larger time steps are used. From [22].
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Diffusion Inspired Shrinkage Functions

So Far  connections between soft Haar wavelet shrinkage
Now    connections between arbitrary diffusivities and Haar wavelet shrinkage with general shrinkage functions

Therefore:  fully discretisation (space and time)

Remember once more :) the discretisation scheme

\[
\frac{u_i^{k+1} - u_i^k}{\tau} = g(|u_{i+1}^k - u_i^k|)(u_{i+1}^k - u_i^k) - g(|u_i^k - u_{i-1}^k|)(u_i^k - u_{i-1}^k)
\]
Diffusion Inspired Shrinkage Functions

Starting with \( u^0_i = f_i \) we obtain

\[
\begin{align*}
    u^1_i &= u_i = \frac{f_{i-1} + 2f_i + f_{i+1}}{4} + (f_i - f_{i+1})\left(\frac{1}{4} - tg(|f_i - f_{i+1}|)\right) \\
    &\quad - (f_{i-1} - f_i)\left(\frac{1}{4} - tg(|f_{i-1} - f_i|)\right)
\end{align*}
\]

as first iteration step

On the other hand: translation invariant soft Haar wavelet shrinkage

\[
    u_i = \frac{f_{i-1} + 2f_i + f_{i+1}}{4} + \frac{1}{2\sqrt{2}} S_\theta\left(\frac{f_i - f_{i+1}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}} S_\theta\left(\frac{f_{i-1} - f_i}{\sqrt{2}}\right)
\]

What do we see??
Diffusion Inspired Shrinkage Functions

Comparing shows that both methods are equivalent if

$$\frac{\sqrt{2}}{4} S_\theta \left( \frac{s}{\sqrt{2}} \right) = s \left( \frac{1}{4} - \tau g(|s|) \right)$$

explicit correspondences by

$$S_\theta(x) = x(1 - 4tg(\sqrt{2}|x|))$$

$$g(|x|) = \frac{1}{4t} - \frac{\sqrt{2}}{4tx} S_\theta \left( \frac{x}{\sqrt{2}} \right)$$
Diffusion Inspired Shrinkage Functions

Top: Four popular shrinkage functions: soft, garrote, firm, and hard shrinkage
Bottom: Corresponding diffusivities. [3].
Diffusion Inspired Shrinkage Functions

Top: Four popular diffusivities: linear, Charbonnier, Perona-Malik, and Weickert diffusivity.
Bottom: Corresponding shrinkage functions.
Wavelet Shrinkage with improved Rotation Invariance

• From 1-D signals to 2-D grayscale images
• 2-D Haar wavelet transform
  – Lowpass filter \( L (1/\sqrt{2}, 1/\sqrt{2}) \)
  – Highpass filter \( H (1/\sqrt{2}, -1/\sqrt{2}) \)

\[
\begin{align*}
v_{l+1} & = L(x) * L(y) * v^l, \\
w_{y}^{l+1} & = L(x) * H(y) * v^l, \\
w_{x}^{l+1} & = H(x) * L(y) * v^l, \\
w_{xy}^{l+1} & = H(x) * H(y) * v^l
\end{align*}
\]

– Shrink all coefficients \( \omega_x, \omega_y, \omega_{xy} \) separately
Wavelet Shrinkage with improved Rotation Invariance
Wavelet Shrinkage with improved Rotation Invariance

- Isotropic variant of scalar-valued diffusivity

\[ u_t = \text{div}(g(|\nabla u|) \nabla u) \]

- Derive coupled shrinkage rules

\[
S(w_x) = w_x \left( 1 - 4 \tau g \left( \sqrt{w_x^2 + w_y^2 + 2w_{xy}^2} \right) \right)
\]

\[
S(w_y) = w_y \left( 1 - 4 \tau g \left( \sqrt{w_x^2 + w_y^2 + 2w_{xy}^2} \right) \right)
\]

\[
S(w_{xy}) = w_{xy} \left( 1 - 4 \tau g \left( \sqrt{w_x^2 + w_y^2 + 2w_{xy}^2} \right) \right)
\]
Wavelet Shrinkage with improved Rotation Invariance
Shrinkage of Colour Images

- Inspired by diffusion

\[ \partial_t u_i = \text{div} \left( g \left( \left( \sum_{j=1}^{3} |\nabla u_j|^2 \right)^{1/2} \right) \nabla u_i \right) \]

- Coupling of channels
- Synchronised channels
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Wavelets with Higher Vanishing Moments

- Wavelets with $m \geq 1$ vanishing moments
- $m^{\text{th}}$ order derivatives

\[
S_{\theta}\left(\frac{\gamma_m h^m}{m!} s \right) = s\left(\frac{\gamma_m h^m}{m!} + 2\tau \frac{(-1)^m m!}{\gamma_m h^m} g(|s|)\right)
\]

for $m = 2$

\[
S_{\theta}\left(\frac{\sqrt{3}}{2\sqrt{2}} s \right) = s\left(\frac{\sqrt{3}}{2\sqrt{2}} + 2\tau \frac{2\sqrt{2}}{\sqrt{3}} g(|s|)\right)
\]
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Summary

• Showed connections between wavelet shrinkage and nonlinear diffusion filtering for discrete
• Wavelet-inspired scheme for TV diffusion
• Relation between shrinkage functions and diffusivity
• Shrinkage-functions inspired by diffusion
• Coupling strategies for color images
• Build hybrid schemes
Relations to „Mathematical Fundamentals of Vision and Grafic“

What have we seen in the course?

-> Diffusion Equation
-> Diffusion to reduce noise but preserve edges - 3 approaches to make the diffusion locally adaptive to structure of image
  - Curve evolution approach
  - Energy minimizing approach
  - Non linear PDE approach
    e.g. Perona – Malik equation
    or Total Variation (TV)
References


